

# Seifert surgery on knots via Reidemeister torsion and Casson-Walker-Lescop invariant II

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## Abstract

For a knot  $K$  with  $\Delta_K(t) \doteq t^2 - 3t + 1$  in a homology 3-sphere, let  $M$  be the result of  $2/q$ -surgery on  $K$ . We show that an appropriate assumption on the Reidemeister torsion of the universal abelian covering of  $M$  implies  $q = \pm 1$ , if  $M$  is a Seifert fibered space.

## 1 Introduction

The first author [Kd1] studied the Reidemeister torsion of Seifert fibered homology lens spaces, and showed the following:

**Theorem 1.1** ([Kd1, Theorem 1.4]) *Let  $K$  be a knot in a homology 3-sphere  $\Sigma$  such that the Alexander polynomial of  $K$  is  $t^2 - 3t + 1$ . The only surgeries on  $K$  that may produce a Seifert fibered space with base  $S^2$  and with  $H_1 \neq \{0\}, \mathbb{Z}$  have coefficients  $2/q$  and  $3/q$ , and produce Seifert fibered space with three singular fibers. Moreover (1) if the coefficient is  $2/q$ , then the set of multiplicities is  $\{2\alpha, 2\beta, 5\}$  where  $\gcd(\alpha, \beta) = 1$ , and (2) if the coefficient is  $3/q$ , then the set of multiplicities is  $\{3\alpha, 3\beta, 4\}$  where  $\gcd(\alpha, \beta) = 1$ .*

It is conjectured that Seifert surgeries on non-trivial knots are integral (except some cases). We [KMS] have studied the  $2/q$ -Seifert surgery, one of the remaining cases of the above theorem, by applying the Reidemeister torsion and the Casson-Walker-Lescop invariant, and have given sufficient conditions to determine the integrality of  $2/q$  ([KMS, Theorems 2.1, 2.3]).

In this paper, we give another condition for the integrality of  $2/q$  (Theorem 2.1). Like as in [KMS], the condition is also suggested by computations for the figure eight knot ([KMS, Example 2.2]).

We note two differences of this paper from [KMS]; one is that the surgery coefficient appears in the condition instead of the Casson-Walker-Lescop invariant, and another is that we need more delicate estimation for the Dedekind sum to prove the result.

(1) Let  $\Sigma$  be a homology 3-sphere, and let  $K$  be a knot in  $\Sigma$ . Then  $\Delta_K(t)$  denotes the Alexander polynomial of  $K$ , and  $\Sigma(K; p/r)$  denotes the result of  $p/r$ -surgery on  $K$ .

(2) The first author [Kd2] introduced the norm of polynomials and homology lens spaces: Let  $\zeta_d$  be a primitive  $d$ -th root of unity. For an element  $\alpha$  of  $\mathbb{Q}(\zeta_d)$ ,  $N_d(\alpha)$  denotes the norm of  $\alpha$

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associated to the algebraic extension  $\mathbb{Q}(\zeta_d)$  over  $\mathbb{Q}$ . Let  $f(t)$  be a Laurent polynomial over  $\mathbb{Z}$ . We define  $|f(t)|_d$  by

$$|f(t)|_d = |N_d(f(\zeta_d))| = \left| \prod_{i \in (\mathbb{Z}/d\mathbb{Z})^\times} f(\zeta_d^i) \right|.$$

Let  $X$  be a homology lens space with  $H_1(X) \cong \mathbb{Z}/p\mathbb{Z}$ . Then there exists a knot  $K$  in a homology 3-sphere  $\Sigma$  such that  $X = \Sigma(K; p/r)$  ([BL, Lemma 2.1]). We define  $|X|_d$  by

$$|X|_d = |\Delta_K(t)|_d,$$

where  $d$  is a divisor of  $p$ . Then  $|X|_d$  is a topological invariant of  $X$  (Refer to [Kd2] for details).

(3) Let  $X$  be a closed oriented 3-manifold. Then  $\lambda(X)$  denotes the Lescop invariant of  $X$  ([Le]). Note that  $\lambda(S^3) = 0$ .

## 2 Result

Let  $K$  be a knot in a homology 3-sphere  $\Sigma$ . Let  $M$  be the result of  $2/q$ -surgery on  $K$ :  $M = \Sigma(K; 2/q)$ . Let  $\pi : X \rightarrow M$  be the universal abelian covering of  $M$  (i.e. the covering associated to  $\text{Ker}(\pi_1(M) \rightarrow H_1(M))$ ). Since  $H_1(M) \cong \mathbb{Z}/2\mathbb{Z}$ ,  $\pi$  is the 2-fold unbranched covering.

In [KMS], we have defined  $|K|_{(q,d)}$  by the following formula, if  $|X|_d$  is defined:

$$|K|_{(q,d)} := |X|_d.$$

Assume that the Alexander polynomial of  $K$  is  $t^2 - 3t + 1$ . Then, as noted in [KMS],  $H_1(X) \cong \mathbb{Z}/5\mathbb{Z}$  and  $|K|_{(q,5)}$  is defined.

We then have the following.

**Theorem 2.1** *Let  $K$  be a knot in a homology 3-sphere  $\Sigma$ . We assume the following.*

(2.1)  $\lambda(\Sigma) = 0$ ,

(2.2)  $\Delta_K(t) \doteq t^2 - 3t + 1$ ,

(2.3)  $|q| \geq 3$ ,

(2.4)  $\sqrt{|K|_{(q,5)}} > 4q^2$ .

*Then  $M = \Sigma(K; 2/q)$  is not a Seifert fibered space.*

**Remark 2.2** Let  $K$  be the figure eight knot in  $S^3$ . Note that  $\Delta_K(t) \doteq t^2 - 3t + 1$ . Then  $|K|_{(q,5)} = (5q^2 - 1)^2$  by [KMS, Example 2.2]. Hence (2.4) holds if  $|q| \geq 3$ .

**Remark 2.3** Theorem 2.1 seems to suggest studying the asymptotic behavior of  $|K|_{(q,d)}$  as a function of  $q$ .

### 3 An inequality for the Dedekind sum

To prove Theorem 2.1, we need the following inequality for the Dedekind sum  $s(\cdot, \cdot)$  ([RG]):

**Proposition 3.1** ([Ma, Lemma 3]) *For an even integer  $p \geq 8$  and for an odd integer  $q$  such that  $3 \leq q \leq p-3$  and  $\gcd(p, q) = 1$ , we have*

$$|s(q, p)| < f(2, p)$$

$$\text{where } f(2, p) = \frac{(p-1)(p-5)}{24p}.$$

By this proposition, we immediately have the following.

**Lemma 3.2** *For an even integer  $p \geq 8$  and for an integer  $q_*$  such that  $q_* \not\equiv \pm 1 \pmod{p}$  and  $\gcd(p, q_*) = 1$ , we have*

$$|s(q_*, p)| < \frac{p}{24}.$$

**Proof.** By assumptions, there exists  $q$  such that  $q_* \equiv q \pmod{p}$  and  $3 \leq q \leq p-3$ . Hence by Proposition 3.1, we have

$$|s(q_*, p)| = |s(q, p)| < \frac{(p-1)(p-5)}{24p} < \frac{p}{24}.$$

□

**Remark 3.3** The estimation given in Proposition 3.1 has a natural application ([Ma]).

### 4 Proof of Theorem 2.1

Suppose that  $M = \Sigma(K; 2/q)$  is a Seifert fibered space. Then, as shown in [KMS], we may assume that

(\*) :  $M$  has a framed link presentation as in Figure 1,

where  $1 \leq \alpha < \beta$  and  $\gcd(\alpha, \beta) = 1$ .

$$M = \left( \begin{array}{c} \text{---} \overbrace{\left( \begin{array}{c} K_1 \\ \text{---} \end{array} \right)}^{\text{---}} \overbrace{\left( \begin{array}{c} K_2 \\ \text{---} \end{array} \right)}^{\text{---}} \overbrace{\left( \begin{array}{c} K_3 \\ \text{---} \end{array} \right)}^{\text{---}} \text{---} \end{array} \right)^J_0$$

$\frac{2\alpha}{q_1}$

$\frac{2\beta}{q_2}$

$\frac{5}{q_3}$

Figure 1: A framed link presentation of  $M = \Sigma(K; 2/q)$

Also as shown in [KMS],  $\sqrt{|K|_{(q,5)}} = (\alpha\beta)^2$ . Hence by (2.4),

$$(\alpha\beta)^2 > 4q^2 \quad (4.1)$$

By (2.1), (2.2) and [Le, 1.5 T2], we have  $\lambda(M) = -q$ . Hence  $(\alpha\beta)^2 > 4\{\lambda(M)\}^2$ , and hence

$$|\lambda(M)| < \frac{\alpha\beta}{2} \quad (4.2)$$

We now consider  $e$  defined as follows:

$$e := \frac{q_1}{2\alpha} + \frac{q_2}{2\beta} + \frac{q_3}{5}.$$

According to the sign of  $e$ , we treat two cases separately: We first consider the case  $e > 0$ . Then the order of  $H_1(M)$  is  $20\alpha\beta e$ . Since  $H_1(M) \cong \mathbb{Z}/2\mathbb{Z}$ ,  $20\alpha\beta e = 2$ , and  $e = 1/(10\alpha\beta)$ . Hence by (\*) and [Le, Proposition 6.1.1], we have

$$\lambda(M) = \left(-\frac{4}{5}\right)\alpha\beta + \frac{5\beta}{24\alpha} + \frac{5\alpha}{24\beta} + \frac{1}{120\alpha\beta} - \frac{1}{4} - T \quad (4.3)$$

where  $T = s(q_1, 2\alpha) + s(q_2, 2\beta) + s(q_3, 5)$ .

By (4.2), we have

$$-\frac{\alpha\beta}{2} < \lambda(M).$$

Hence by (4.3),

$$-\frac{\alpha\beta}{2} < \left(-\frac{4}{5}\right)\alpha\beta + \frac{5\beta}{24\alpha} + \frac{5\alpha}{24\beta} + \frac{1}{120\alpha\beta} - \frac{1}{4} + |T|.$$

Consequently

$$\frac{3}{10}\alpha\beta < -\frac{1}{4} + \frac{5}{24\alpha}\beta + \frac{5}{24}\left(\frac{\alpha}{\beta}\right) + \frac{1}{120\alpha\beta} + |T| \quad (4.4)$$

As in [KMS], we show that  $\alpha \geq 2$  implies a contradiction: Suppose that  $\alpha \geq 2$ . Since  $\alpha < \beta$ , we have  $\beta \geq 3$  and  $\alpha/\beta < 1$ . Hence

$$\frac{3}{5}\beta < -\frac{1}{4} + \frac{5}{24 \cdot 2}\beta + \frac{5}{24} + \frac{1}{120 \cdot 2 \cdot 3} + |T|.$$

Since  $|s(q_1, 2\alpha)| \leq \frac{2\alpha}{12} < \frac{2\beta}{12}$ ,  $|s(q_2, 2\beta)| \leq \frac{2\beta}{12}$ , and  $|s(q_3, 5)| \leq \frac{1}{5}$  as in [KMS], we have

$$|T| \leq |s(q_1, 2\alpha)| + |s(q_2, 2\beta)| + |s(q_3, 5)| \leq \frac{\beta}{3} + \frac{1}{5}.$$

Hence

$$\frac{3}{5}\beta < -\frac{1}{4} + \frac{5}{48}\beta + \frac{5}{24} + \frac{1}{120 \cdot 6} + \left(\frac{\beta}{3} + \frac{1}{5}\right).$$

Thus

$$\left(\frac{3}{5} - \frac{5}{48} - \frac{1}{3}\right)\beta < -\frac{1}{4} + \frac{5}{24} + \frac{1}{120 \cdot 6} + \frac{1}{5}.$$

Therefore

$$\frac{39}{240}\beta < \frac{1}{240} \left(38 + \frac{1}{3}\right) < \frac{39}{240}.$$

This contradicts  $\beta \geq 3$ .

We next show that  $\alpha = 1$  implies a contradiction: Suppose that  $\alpha = 1$ . By (4.1),  $\beta^2 > 4q^2$ . Since  $|q| \geq 3$ ,  $\beta^2 > 4 \cdot 3^2 = 36$ . Hence  $\beta > 6$ . Since  $\alpha = 1$ ,  $e = \frac{1}{10\beta}$ . Hence

$$\frac{q_1}{2} + \frac{q_2}{2\beta} + \frac{q_3}{5} = \frac{1}{10\beta}$$

and hence we have the following equation.

$$(5\beta)q_1 + 5q_2 + (2\beta)q_3 = 1 \quad (4.5)$$

Since  $q_1$  and  $q_2$  are odd (see Figure 1),  $\beta$  must be even. Since  $\beta > 6$ , we have  $\beta \geq 8$ . We then have

$$(\#): \quad q_2 \not\equiv \pm 1 \pmod{2\beta}.$$

In fact, since  $q_1$  is odd,  $(5\beta)q_1 \equiv \beta \pmod{2\beta}$ . Hence by (4.5),

$$\beta + 5q_2 \equiv 1 \pmod{2\beta}.$$

Now suppose that  $q_2 \equiv 1 \pmod{2\beta}$ . Then  $\beta + 5 \equiv 1 \pmod{2\beta}$ . This is impossible since  $\beta \geq 8$ . Next suppose that  $q_2 \equiv -1 \pmod{2\beta}$ . Then  $\beta - 5 \equiv 1 \pmod{2\beta}$ . This is also impossible since  $\beta \geq 8$ . Thus  $(\#)$  holds.

Substituting  $\alpha = 1$  in (4.4),

$$\frac{3}{10}\beta < -\frac{1}{4} + \frac{5}{24}\beta + \frac{5}{24\beta} + \frac{1}{120\beta} + |T|$$

where  $T = s(q_2, 2\beta) + s(q_3, 5)$  (since  $s(q_1, 2) = 0$ ). By  $(\#)$  and Lemma 3.2,

$$|s(q_2, 2\beta)| < \frac{2\beta}{24} = \frac{\beta}{12}.$$

Hence

$$|T| \leq |s(q_2, 2\beta)| + |s(q_3, 5)| < \frac{\beta}{12} + \frac{1}{5}.$$

Since  $\beta \geq 8$ ,

$$\frac{3}{10}\beta < -\frac{1}{4} + \frac{5}{24}\beta + \frac{5}{24 \cdot 8} + \frac{1}{120 \cdot 8} + \left(\frac{\beta}{12} + \frac{1}{5}\right).$$

Thus

$$\left(\frac{3}{10} - \frac{5}{24} - \frac{1}{12}\right)\beta < -\frac{1}{4} + \frac{5}{24 \cdot 8} + \frac{1}{120 \cdot 8} + \frac{1}{5}$$

and hence  $\frac{1}{120}\beta < 0$ . This is a contradiction, and ends the proof in the case  $e > 0$ .

We finally consider the case  $e < 0$ . Then  $e = -\frac{1}{10\alpha\beta}$ . By  $(*)$  and [Le, Proposition 6.1.1], we have

$$\lambda(M) = -\left\{\left(-\frac{4}{5}\right)\alpha\beta + \frac{5\beta}{24\alpha} + \frac{5\alpha}{24\beta} + \frac{1}{120\alpha\beta} - \frac{1}{4} + T\right\}.$$

Remaining part of the proof is similar to that in the case  $e > 0$ .

This completes the proof of Theorem 2.1. □

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